

Topology Qual, Algebraic Topology:  
Summer 2012

- (1) Let  $\Sigma_g$  denote the closed, orientable, surface of genus  $g$ .  
Prove that if  $\Sigma_g$  is a covering space of  $\Sigma_h$ , then there is a  $d \in \mathbb{Z}^+$  satisfying

$$g = d(h - 1) + 1.$$

- (2) Let  $X$  be a closed (i.e., compact & boundaryless), orientable  $n$ -dimensional manifold. Prove that if  $H_{k-1}(X; \mathbb{Z})$  is torsion-free, then so is  $H_k(X; \mathbb{Z})$ .

- (3) Let  $T^2 = \mathbb{R}^2/\mathbb{Z}^2$  be the 2-torus, concretely identified as the quotient space of the Euclidean plane by the standard integer lattice. Then any  $2 \times 2$  integer matrix  $A$  induces a map

$$\phi : (\mathbb{R}/\mathbb{Z})^2 \rightarrow (\mathbb{R}/\mathbb{Z})^2$$

by left (matrix) multiplication.

- (a) Show that (with respect to a suitable basis) the induced contravariant map

$$\phi^* : H^1(T^2; \mathbb{Z}) \rightarrow H^1(T^2; \mathbb{Z})$$

on the cellular cohomology is left multiplication by the transpose of  $A$ .

- (b) Since  $T^2$  is a closed, oriented manifold, it has a fundamental class,  $[T^2] \in H_2(T^2; \mathbb{Z})$ . Prove that

$$\phi_*[T^2] = \det(A) [T^2].$$

(Hint: Use part (a) and the naturality of the cup product under induced maps on homology/cohomology.)

- (4) The closed, orientable surface  $\Sigma_g$  of genus  $g$ , embedded in  $\mathbb{R}^3$  in the standard way, bounds a compact region  $R$  (often called a genus  $g$  solid handlebody).

Two copies of  $R$ , glued together by the identity map between their boundary surfaces, form a closed 3-manifold  $X$ . Compute  $H_*(X; \mathbb{Z})$ .

## GT Qual 2012 (Spring) Part II

**Show All Relevant Work!**

1) Consider stereographic projection of the unit circle  $S^1$  in  $\mathbf{R}^2$  to  $\mathbf{R}$  from the North Pole  $(0, 1)$  and from the South Pole  $(0, -1)$ .

a) Show that  $\sigma^{-1}(x) = \frac{1-x^2}{1+x^2}$

b) Consider the smooth vector field  $\frac{d}{dx}$  on  $\mathbf{R}$ . Using  $\sigma$ , this induces a smooth vector field on the circle minus the North Pole. Can it be extended to a smooth vector field on all of  $S^1$ ?

2a) A smooth map  $F : M \rightarrow N$  is a *submersion* if...

b) Let  $M$  be a compact, smooth 3-manifold. Prove that there is no submersion  $F : M \rightarrow \mathbf{R}^3$ .

3) Consider  $D$  the open unit disk in  $\mathbf{R}^2$  with Riemannian metric

$$g = \left(\frac{2}{1+x^2+y^2}\right)^2 dx^2 + \left(\frac{2}{1+x^2+y^2}\right)^2 dy^2$$

a) Write down an (oriented) orthonormal frame  $(E_1; E_2)$  for  $D$  with respect to this metric.

b) Write down the associated dual coframe  $(\theta^1; \theta^2)$ .

c) Compute  $\int_D \theta^1 \wedge \theta^2$ . Is this the Riemannian volume form (that is, does it agree with the volume formula  $\sqrt{\det(g_{ij})} dx \wedge dy$ )?

d) Compute the volume (area?) of  $D$  with respect to this metric.

e) What have you computed?

4) Suppose that  $f_0$  and  $f_1$  are smoothly homotopic maps from  $X$  to  $Y$  and that  $X$  is a