Real Analysis Qualifying Exam

Answer all four questions. In your proofs, you may use any major theorem, except the result you are trying to prove (or a variant of it). State clearly what theorems you use. All four questions are worth the same number of points. Good luck.

Question 1. Let $f : [0,1] \to \mathbb{R}$ be a nonnegative Lebesgue measurable function such that $f > 0$ almost everywhere. Prove that for any > 0, there exists > 0 such that for any Lebesgue measurable subset $S \quad [0,1]$ with $m(S) >$, we have $\int_S f \, dm >$

Question 4. Let $(X; ||\ \ ||)$ be a normed ($\mathbb{R}$-)linear space and let $(X^*; ||\ \ ||_{op})$ denote its dual Banach space of (real-valued) bounded linear functions (equipped with the operator norm). Prove that the linear map $: X \to X^{**}$ given by

$$(x)(f) = f(x)$$

is an isometry.

(You may use without proof the fact that for each $x \in X$ there exists $f \in X^*$ such that $||f||_{op} = 1$ and $||x|| = f(x)$.)