ALGEBRA QUALIFYING EXAM – SPRING 2017

Problem 1. Prove that an Artinian ring has finitely many maximal ideals.

Problem 2. Let $\mathbb{F}$ be a finite field with $|\mathbb{F}| = q$. Consider the subgroup

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{F}^{\times}, b \in \mathbb{F} \right\} < \mathrm{GL}_2(\mathbb{F}).$$

Show that for any prime p dividing $q - 1$, the number of Sylow p-subgroups of G is q.

Problem 3. Let R be a UFD and a, b be coprime elements in R. For all $i \geq 0$, compute

$$\mathrm{Tor}_i^{R/(ab)}(R/(a), R/(b)).$$

Problem 4. Let F be a field, and D be an integral domain containing F. Suppose D is finite dimensional as a vector space over F. For each $x \in D$, define the F-linear transformation $T_x : D \to D$ by $T_x(y) = xy$.

(a) Prove that F